Chebychev series for calculating Arctangent

1 Preliminaries

We make use of the following trigonometric relations

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \tag{1}$$

$$A + B = \tan^{-1}\left[\frac{\tan A + \tan B}{1 - \tan A \tan B}\right] \tag{2}$$

and we find the quadrature

$$I = \int_0^1 \frac{\tan^{-1} x}{x} \mathrm{T}_0(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} \tag{3}$$

algebraically. In eq (3) we substitute $x = 2t/(1+t^2)$ to find

$$I = \int_0^1 (\tan^{-1} Pt + \tan^{-1} Qt) \frac{\mathrm{d}t}{t} \tag{4}$$

where P and Q are roots of $\alpha^2 - 2\alpha - 1 = 0$. Writing u for Pt and then Qt and then replacing u by $1/u$ we find

$$I = \frac{\pi}{2} \ln \frac{1}{q} \tag{5}$$

where q is the postive root of $q^2 + 2q - 1 = 0$.

$$\int_0^1 \mathrm{T}_n(x)\mathrm{T}_n(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{\pi}{4}(1 + \delta_{n.0}) \tag{6}$$

where $\delta_{n,m}$ is the Kronecker delta symbol (=1 if $n = m$ and =0 otherwise).

The arctangent $\tan^{-1}(x)$ can be represented as a series of odd order Chebychev polynomials in x or as x times a series of even order Chebychev polynomials in x. We consider both.

2 Odd expansion of Arctangent

Determination of the odd series coefficients can be done by a relatively well known trick. The odd series expansion should become identical to the Maclaurin expansion when the number of terms is infinite. From de Moivre's theorem $\mathrm{T}_{2n+1}(x) = \frac{1}{2}[(x+iy)^{2n+1} + (x-iy)^{2n+1}]$ where $x = \cos\theta$ $y = \sin\theta$. Hence $\sum_0^\infty k^{2n+1}\mathrm{T}_{2n+1}(x)$ is the sum of the $(2n+1)^{\mathrm{th}}$ power coefficients in the series for $\frac{1}{2}[\tan^{-1} k(x+iy) + \tan^{-1} k(x-iy)]$ with multipliers $\frac{(-1)^n}{2n+1}$. This sum is $\frac{1}{2}\tan^{-1}[2xk/(1-k^2)]$ which is equal to $\frac{1}{2}\tan^{-1} x$ if $k^2+2k-1=0$. For the Chebychev expansion to converge the root of magnitude smaller than 1 must be chosen; this is q as defined above. Hence the odd series with $N+1$ terms (up to T_{2N+1}) is

$$\tan^{-1} x \approx P_O(x) = \sum_0^N a_n \mathrm{T}_{2n+1}(x) \tag{7}$$

where $a_n = \frac{2}{2n+1} q^{2n+1}(-1)^n$.

3 Even expansion of Arctangent

The expansion given by x times an even series with $N+1$ terms (up to T_{2N}) is

$$\tan^{-1} x \approx P_E(x) = x\sum_1^N b_n \mathrm{T}_{2n}(x) + \frac{x}{2} b_0 \mathrm{T}_0(x) \tag{8}$$

where the coefficients b_n are to be determined; we need to relate them to a_n. From the recurrence relations for Chebychev polynomials we find

$$(1+E)b_n = 2a_n \tag{9}$$

where E is a shift operator that increases the subscript by 1.

$$b_n = \frac{2}{1+E} a_n + K \tag{10}$$

where K is an arbitrary constant and the inverse of $1-E$ is interpreted by the binomial theorem. Comparison of eq (5) and(10) with $n=0$ shows that $K=0$. From these equations we find

$$P_E(x) - P_O(x) = -[(1+E)^{-1} a_{N+1}]\mathrm{T}_{2N+1}(x) \tag{11}$$

$$P_E(x) - P_O(x) = 2(-1)^{-N}\left[\frac{q^{2N+3}}{2N+3} + \frac{q^{2N+5}}{2N+5} + \dots\right]\mathrm{T}_{2N+1}(x) \tag{12}$$

4 Coefficients of powers of x

4.1 Odd series

The coefficient of x^{2j+1} in $\mathrm{T}_{2m+1}(x)$ is

$$G_{j,m} = 2^{2j}\frac{2m+1}{2j+1}\left[{}^{m+j}\mathrm{C}_{m-j}\right](-1)^{m-j} \tag{13}$$

where C denotes a binomial coefficient. The coefficient of x^{2j+1} in the odd series expansion is $\sum_{m=j}^{N} G_{j,m}a_m$. With the expression already found for a_m, the equation $q^2+2q-1=0$ satisfied by q and the substitution $k=m-j$ in this sum we find that D_j is the factor $\frac{(-1)^j}{2j+1}(1-q^2)^{2j+1}$ multiplied by the sum of the first $N+1-j$ terms of $(1-q^2)^{-2j-1}$ expressed as a power series in q^2.

4.2 Even series

We get the powers of the even series from eq (12) relating the odd and even expansions.

We add $(-1)^j 2^{2j+1}\frac{2N+1}{2j+1}\left[{}^{N+j}\mathrm{C}_{N-j}\right]\left[\frac{q^{2N+3}}{2N+3}+\ldots\right]$ to D_j. The expression inthe second pair of square brackets is one half of the difference between $\ln\frac{1}{q}$ and the first $N+1$ terms in the expansion of $\ln\frac{1+q}{1-q}$ as a series in q. However this expression and the difference alluded to for the powers in the odd expansion should not be used to find the coefficients numerically to avoid cancellation.